Synthesis-Algorithm Of Bang-bang Controller With Delayed Feedback On Temperature Controlled Systems

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Abstract—A bang-bang controller offers some advantages, such as simple structure and low cost. However, this controller always provides signals in some manipulated variables in fluctuating values. In order to improve its characteristic, the bang-bang controller is connected with a delayed feedback as a compensator. This paper proposed a synthesis algorithm for determining relationship between parameters of bang-bang controller with delayed feedback to parameters of temperature controlled systems as plant. The delayed feedback component is a second order filter which has two time constants and a gain steady state parameter. This Bang-bang controller with delayed feedback of second order filter can be assumed as non-linear PID controller with low pass filter.

Keywords—Bang-bang controller, Temperature Controlled Systems.

I. INTRODUCTION

A Bang-bang controller with two output conditions is classified as the discontinuous controllers [1]. It is employed for specific applications that have relatively low performances on response quality. The conventional bang-bang controllers do not require any external energy for their functionality, such as a bimetal or manual switch which are sometimes consist of two components, such as temperature sensor and bang-bang controller at the same time. It is commonly employed in many home appliances as a temperature controller. Compared with continuous controllers, these controllers offer some advantages, namely simple construction, low cost, and good reliability. [3]

The fundamental of bang-bang controllers is, that output controller variable is not continued, but it can only provide a certain value. In related to the two-position controllers, the Output Controller Variable (OCV) always move between two values e.g. high and low signals, open and close switch or valve, on and off state [3].

The difference between low-high level is so-called hysteresis. Manipulated variable preserves its previous state between turn on and turn off levels (variable logic). Bang-bang control system is a special type of control method where control move is only allowed to be discrete value, which is very similar to operators’ control actions. Therefore, it is natural to set up a rough set controller which is combination between Bang-bang controller with delayed feedback component as non-linear PID controller. In the whole industry process [4], the generalized Bang-bang controller can be defined as controller with the output signals [5] in only two condition, saturated input will be generated saturated output.

If the value of difference between reference and controlled variable is \( x_3 < y_H \), the manipulated variable has a value equal to 1, and if the value is \( x_3 < y_H \), the manipulated variable is 0. In bang-bang control, the regulator and the plant cooperate without modification from a source. The required value is reached by cyclic switching of the manipulated variable. The variable is influenced by the lag in a whole system, which causes an overshoot situation [6]. The value of controlled variable \( y \) does not respond to the change of an action variable \( u \) but after a delay. This phenomenon has a bad impact on quality of control process. This circumstance can be suppressed or even eliminated with proper control logic. The basic principle of the on-off controller is defined by the input control algorithm [7].

II. BANG-BANG CONTROLLER STRUCTURES

Based on configuration of controller algorithm, are that bang-bang controller can be classified into two kind of algorithms: (1) Conventional algorithm and (2) fix-frequency Bang-bang controllers or bang-bang controller with delayed feedback algorithm. Fig. 1 shows the symbol of Bang-bang controller with two conditions. Controller output will be high, if input signal moves to value of bigger than \( \varepsilon \). It will be low condition if input is smaller than \(-\varepsilon\).

![Fig. 1 Conventional Bang-bang Controller](image)

III. CONVENTIONAL MODE

Based on feedback structure in controller path, bang-bang controller can be classified into 4 algorithms. However, in this paper, will be described only two structures of Bang-bang controller: Conventional Mode and PWM Mode. Fig. 2...
illustrates the Bang-bang controller which is arranged in conventional mode. If the output of controlled system $x_2(t)$ is lower than the reference value $x_1(t)$ (or more than difference between $x_1(t) - \varepsilon$, and if hysteresis exist), then the manipulated variable $y(t)$ will be high condition. This state is commonly known as a power mode. When the controlled variable reaches the set-point (or meets the expression $x_3(t) = 0$, the control logic will turn the controller off. This phase is called the phase of load.

The controller is turned on again when the controlled variable falls under the set-point (or the expression $(x_3(t) - \varepsilon) < 0$) This complete signal is for the first period of the switching cycle. The controlled variable $x_3(t)$ is maintained at the desired value by this periodic operation. If it is assumed that temperature controlled system can be approximated into the second differential equation, mathematically the dynamic of controlled system can be represented what happens during the phase of power supply and phase of load, a dynamic of a stable system of a higher order was investigated. Its differential equation is in form as follows:

$$a_2 \frac{d^2x_3}{dt^2} + a_1 \frac{dx_3}{dt} + a_0x_3(t) = b_0y(t)$$  \hspace{1cm} (1)

The $x_3(t)$ represents the controlled output (variable of controlled temperature system) and $y(t)$ represents the system’s input. The equation of both phases can be derived from (1). Since the phase of the power supply is characterized as a response of a system to a high state, the equation defining this phase will be a complete solution of (1), with considered initial conditions. The other phase, the phase of load, is the reaction of a system to a low state of a controller, thus the equation for this state will be a homogenous solution of (1), with initial conditions.

![Fig. 2 Bang-bang controller in Conventional Mode](image)

Fig. 2 shows structure of closed loop control using bang-bang controller without delayed feedback. The output signal of controller $y(t)$ directly drives input of controlled process (plant). Controlled variable $x_2(t)$ is measured and inserted to the summing point of system, in order to be subtracted to the reference signal $x_1(t)$. The difference between output of closed loop system with reference signal $(x_3(t) = x_1(t) - x_2(t))$ provides an idea of Bang-bang controller to have a low or high condition.

![Fig. 3 Solution of Conventional Mode](image)

**V. RESPONSE OF CONVENTIONAL BANG-BANG CONTROL**

Response of Bang-bang control system can be obtained by finding of solution of equation 1. Its solution can be derived based on its two states which are in power source mode or in load mode. Since the output signals of bang-bang controller whether in high or low state is always in rectangular shape signal (step signal), so the solution of equation 1 in power source mode is in exponential form as shown in equation 2 and 3, where $T_L$ is a dead time and $T_e$ are time constant of controlled temperature systems.

$$x_1 + \varepsilon = x_{2\text{max}} + \left(V_{y01} + z - x_{2\text{min}}\right) \left(1 - e^{\frac{t - t_L}{T_e}}\right) \hspace{1cm} (3)$$

$$x_{2\text{max}} = x_1 + \varepsilon + \left(V_{y01} + z - x_1 - \varepsilon\right) \left(1 - e^{-\frac{t_L}{T_e}}\right) \hspace{1cm} (4)$$

As in Fig. 3, the load mode is represented by exponential decreasing signals which its signal is decreasing exponentially in between $x_{2\text{max}}$ value to $x_{2\text{min}}$ value. Equations 5 and 6 show the mathematical solution of dynamic characteristic as shown by equation 1 for load mode.

$$x_1 - \varepsilon = x_{2\text{max}} + \left(V_{y02} + z - x_{2\text{min}}\right) \left(1 - e^{\frac{t_L - t}{T_e}}\right) \hspace{1cm} (5)$$

$$x_{2\text{min}} = x_1 - \varepsilon + \left(V_{y02} + z - x_1 + \varepsilon\right) \left(1 - e^{\frac{t_L}{T_e}}\right) \hspace{1cm} (6)$$

The calculation of $D$(duty cycle) and $f_{SW}$ (switch frequency) of bang-bang controller output is easily based on curves shown by Fig. 3. Parameters $D$ and $f_{SW}$ can be founded by knowledge of both time parameters $t_1$ and $t_2$. The first time variable $t_1$ and $t_2$ is directed by derived curve output respect to time. The both equation are shown in equation 7 and 8.

![Fig. 3 Solution of Conventional Mode](image)
Where:
\[
t_1 \approx T_e \frac{2\Delta t}{\gamma_{y_{02}}-x_1} \tag{7b}
\]

Analogy to the equation 6 and 7, second time variable \( t_2 \) can be calculated by the gradient equation shown by equation 8.

\[
\frac{dx_2}{dt} \approx \frac{\gamma_{y_{02}}-x_1}{T_e} < 0 \tag{8}
\]

Based on equation 8, the second duration-time variable is formulated as shown by equation 9.

\[
t_2 \approx T_e \frac{2\Delta t}{x_1-\gamma_{y_{02}}} \tag{9}
\]

According to both equations 7 and 9, duty cycle (\( D \)) parameter is found by:

\[
D = \frac{t_1}{t_2} \approx \frac{x_1-\gamma_{y_{02}}}{\gamma_{y_{01}}-x_1} \tag{10}
\]

and switch frequency is determined by:

\[
f_{sw} = \frac{1}{t_1+t_2} \approx \frac{1}{T_e} \frac{\gamma_{y_{01}}-x_1}{2\Delta t}\gamma_{y_{01}}-\gamma_{y_{02}}} \tag{11}
\]

Equations 10 and 11 show that duty cycle and switch frequency are depend on strongly to reference signal (desired value or trajectory of closed loop system). If desired value is located exactly in the middle of set-point variable, the output response of system is fully symmetry. For the other value of set-point, the output system response is varied.

VI. PULSE WIDTH MODULATION MODE

The principle of PWM mode is based on structure of controller is different and more complex than that of comparative mode. Its control algorithm is built on existing of feedback path which is depend on functionality between the characteristic of feedback component and hysteresis width. Since the output of bang-bang controller is always in two level, high/low signals, the output of algorithm is in a square-wave waveform and have different duty cycle.

The conventional mode drives the plant input with energy from a power source until the controlled variable \( x_2 \) reaches the desired value. Because of the lag present in the temperature controlled system, the controlled variable \( x_2 \) smaller than the desired value, so this algorithm can reduce quality and precision of the regulation process. The PWM mode bypasses this problem with precisely calculated time constant of feedback component so can generate signal with duty cycle variation and frequency of the control signal. The controlled variable will slowly reach the desired condition without any considerable overshoot. The disadvantage of this mode lies in a longer settling time of closed loop control.

A mathematical model of this type of control algorithm is based on pseudo plant (component on feedback path of bang-bang controller) and real plant (temperature controlled system). A response of a feedback component plays important role in this structure of bang-bang controller, because this component has a functionality as pseudo plant. The solution of this circuit consist of two equations which are involve the power source and the load mode. The solution where represent as power source mode is shown by equation 12 and for load mode is by equation 13.

\[
x_{4_{\text{max}}} = x_3 + \varepsilon = (x_3 - \varepsilon) + (\gamma_{y_{01}} - x_3 + \varepsilon) \left(1 - e^{-\frac{t_1}{T_y}}\right) \tag{12}
\]

\[
x_{4_{\text{min}}} = x_3 - \varepsilon = (x_3 + \varepsilon) + (\gamma_{y_{01}} - x_3 - \varepsilon) \left(1 - e^{-\frac{t_2}{T_y}}\right) \tag{13}
\]

Where the first term is an objective value of the closed loop control system. The second term of this equation represents solution for bang-bang controller with a delayed feedback and the third term is exponential term of solution. For duty cycle and switch frequency parameters are obtained by deriving from both equations (equations 12 and 13). The first time-duration parameter can be determined using the equation 12 and is shown in equation 14.

\[
t_1 = T_y \left\{ \text{ln} \left( \frac{\gamma_{y_{01}} - x_3 + \varepsilon}{\gamma_{y_{01}} - x_3 - \varepsilon} \right) \right\} \tag{14}
\]

Equation 14 shows that duration of power source mode is not depend on the set-point value or trajectory reference of closed loop system. Therefore, it can be concluded that the PWM mode can improve the dynamic characteristic of bang-bang controller especially for linearity of systems. Analog to equation 14, second time-duration parameter is shown by equation 15.

\[
t_2 = T_y \left\{ \text{ln} \left( \frac{\gamma_{y_{02}} - x_3 - \varepsilon}{\gamma_{y_{02}} - x_3 + \varepsilon} \right) \right\} \tag{15}
\]

Duty cycle of output bang-bang controller is obtained by the equation 14 is divided by equation 15 and be multiplied by 100%. This is shown in equation 16.

\[
D = \frac{t_1}{t_1+t_2} \times 100\% \tag{16}
\]

Since both time-duration parameters are not affected by set-point value, so duty cycle of output bang-bang controller is also not influenced by value of reference signals. The other important parameter is switch frequency which can be obtained by:

![Fig. 4 Bang-bang Controller with delayed Feedback](image-url)
If equation 17 is compared with equation 11, on conventional mode, the reference signals influence strongly to output signals of system. However, in the PWM mode, the reference set point does not influence duty cycle and switch frequency of closed loop system with bang-bang controller when a feedback path is added with a second order component (see Fig. 4).

VII. SYNTHESIS-DESIGN ALGORITHM

Synthesis algorithm is started from transfer function of closed loop systems. In order to that output variable of closed loop control can follow the reference signal perfectly, consequently transfer function of closed loop should be equal to one \((C(s)/R(s)) = 1\). Equation 18 shows the Laplace equation for the unity feedback closed loop control.

\[
C(s) \times R(s) = \frac{G(s)}{1 + G(s)/R(s)}
\]

Equation 27 shows a transfer function of closed loop system (plant) and \(G_c(s)\) is controller. The Synthesis algorithm is developed by based on that the output signal can follow the trajectory of reference signal every time, so the transfer function closed loop \(C(s)/R(s) = 1\). If controller in equation 18 is moved to the left of equal sign, so the controller can be found in form of equation 19.

\[
G_c(s) = \frac{\tau_c s + 1}{\tau_c s + 1}
\]

In this paper, it is assumed that response of closed loop system is in first order response. The performance of closed loop system with first order can be figured out by two parameters, duty cycle and switch frequency. In Laplace equation form, the first order component can be represented in two parameters, Gain steady state \((G_{ss})\) and time constant \((\tau_c)\). Equation 20 describes the first order component.

\[
C(s) \times R(s) = \frac{G_{ss}}{\tau_c s + 1}
\]

Equation 21 describes the relationship between performance parameter with the controller. So the controller equation can be obtained using equation 22.

\[
G_c(s) = G(s) \frac{\tau_c s + 1}{\tau_c s + 1}
\]

From equation 22, it can be concluded that if the \(G(s) = 1\), the equation results an integrator controller. Since integral-controller in a closed loop system can provide a zero error steady state (offset is equal zero), so this controller is able to contribute the good performance result design. For temperature controlled system, its dynamic characteristic is commonly approximated by the first order model with dead time. Equation 23 shows the Laplace equation for first order model with dead time.

\[
G(s) = \frac{Ke^{-\tau_D s}}{\tau s + 1}
\]

Since equation 23 consist of a exponential term, therefore for simplicity, the equation should be approximated using Taylor series. Based on that Taylor approximation, the first order model with dead time is converted into the second order model. Equation 24 shows the conversion result.

\[
G(s) = \frac{K}{(\tau z + 1)(\tau z + 1)}
\]

The second order model as shown in equation 24 have three parameters (Gain \(K\), and two time constant : \(\tau\) and \(\tau_D\)). They Fig. out the dynamic characteristic of second order model. This model in Laplace form, have two poles \(p_1 = 1/\tau\); \(p_2 = 1/\tau_D\) and no zero.

If equation 24 is substituted into equation 18 the controller equation is obtained. Equation 25 shows the controller for plant of second order model.

\[
G_c(s) = \frac{\tau z + 1}{K}(\tau z + 1)
\]

Controller equation which is shown by equation 25 have two zeros and 1 pole. Since order of its denominator is smaller than order of its nominator, so transfer function is categorized as un-proper equation and consequently is not realizable. In order to that controller can be implemented in real applications, the controller is added a low pass filter, so that the total transfer function of controller have similar order between denominator and nominator. Equation 26 shows a modification of PID controller as shown by equation 25.

\[
G_c(s) = \frac{r}{K}\left(1 + \frac{1}{\tau_D}\right)(\tau_D s + 1)
\]

Equation 27 shows a PID controller which has been added by low pass filter.

\[
G_c(s) = K_c \left(1 + \frac{1}{\tau_D}\right)(\tau_D s + 1)
\]

Based on equation 27. Parameters of PID controller for controlled temperature systems can be calculated with given plant-parameters and performance. The relationship between PID parameters and given plant parameters is shown by equation 28.

\[
K_c = \frac{r}{K}\tau_D; \tau_i = \tau; \tau_D = \tau_D
\]

Equation 28 are synthesis-design results for temperature controlled system in designing a bang-bang controller with delayed feedback. The relationship between PID controller parameters with gain and time constant in delayed feedback component can be derived based on Fig. 5.
The mathematical relationship between PID-controller parameters with second order parameters is found based on Fig. 5. When the bang-bang controller is in power source mode (on-condition), the Laplace form is shown in equation 29.

\[ G_c(s) = \frac{1}{1 + \frac{T_s \tau_p}{\tau_i} + \frac{T_s}{\tau_d}} \]  

(29)

If it is assumed that:

\[ \frac{T_s \tau_p}{\tau_i} \gg 1 \]  

(30)

So the equation 31 can be changed into a simple equation:

\[ G_c(s) = \frac{(\tau_p \tau_i + 1)}{T_s \tau_d} \]  

(31)

Based on equation 31, PID parameters of bang-bang controller with second order delayed feedback are shown by equation 32. Equation 32 shows synthesis design for bang bang controller.

\[ K_{pc} = \frac{1}{K_r \tau_p} \left( \frac{1 + \tau_p}{\tau_i} \right); \quad K_{ic} = \frac{1}{K_r \tau_i} \quad \text{and} \quad K_{dc} = \frac{T_s}{K_r} \]  

(32)

If between equation 28 are equalled to equation 32, so three parameters of second order delayed feedback are found.

**VIII. RESULTS AND DISCUSSIONS**

Dynamic characteristic of temperature controlled system can be described as first order model with dead time. Equation 33 shows Laplace form of dynamic characteristic of temperature controlled system.

\[ G(s) = \frac{K}{T_s + 1} e^{-t_0 s} \]  

(33)

Where K is gain steady state which it describes the ratio between steady state value of output temperature and electrical current of heater, \( \tau \) is time constant and \( t_0 \) is dead time. Three parameters were obtained by direct measurement in control engineering laboratory at Faculty of Engineering Brawijaya University.

Output temperature was measured and recorded in ARDUINO. The measurement data then was converted into graphic. Fig. 6 shows the graphic which it describes the relationship between temperature output with variable of time. Based on the graphic as shown Fig. 6, value of gain (K), time constant (\( \tau \)), and dead time (\( t_0 \)) are obtained.

Based on Fig. 6, gain steady state, time constant and dead time can be measured directly and result model that is shown in equation 34.

\[ G(s) = \frac{2.5}{85.2 \tau_s + 1} e^{-17.8s} \]  

(34)

Equation 34 consist of exponential term which it lead to difficulty for designing controller, so the equation 38 should be first converted into polynomial Laplace Equation. Easily, dead time and time constant in equation 38 can be converted into first time constant and second time constant. Equation 35 shows the conversion result.

\[ G(s) \approx \frac{2.5}{(85.2 \tau_s + 1)(17.8s + 1)} \]  

(35)

For simplicity of calculation, it is assumed in this verification process that time constant of closed loop system is expected in 95.5 sec. and gain steady state is about one. The equation 36 shows the expected performance of closed loop system.

\[ G_c(s) = \frac{K_c}{95.5s + 1} \]  

(36)

Based on equation 36, expected performance of closed loop system are \( K=1 \) and \( \tau_c = 95.5 \text{ sec} \). Parameters of temperature controlled system are shown by equation 38 and 39 which are \( \tau = 85.2 \text{ sec} \) and dead time \( t_0 = 17.8 \text{ sec} \). If four parameters are substituted into equation 28, parameters of second order delayed feedback can be found. Equation 37, 38 and 39 show the calculation of parameters of the second order delayed feedback.
\[
\frac{\tau}{K_p T_c} = \frac{1}{K_c} \left( \frac{\tau_I + \tau_v}{\tau_I} \right) = \frac{85.2}{0.96(8.52)} = 10.42
\]  
(37)

\[
K_c T_I = \tau = 85.2 \rightarrow \tau_I = \frac{85.2}{K_c}
\]

(38)

\[
\frac{\tau_c}{\tau_v} = t_0 = 17.8 \rightarrow \tau_v = \frac{\tau_c}{17.8}
\]

(39)

If it is assumed that \(K_c = 100\), so both time constant of delayed feedback components are \(\tau_I = 0.852 \text{ sec}\). and \(\tau_v = 0.056 \text{ sec}\).

Fig. 8 Simulink of Bang bang Controller with Delayed Feedback

Fig. 8 shows the bang-bang controller with PWM mode for expected performance as shown by equation 40. Fig. 9 illustrates the controller output signal and error signal of bang-bang controller. The number of measured data was only 218 samples, each samples represents duration of 0.01 second, so totally the duration of signal was 2.18 second.

Fig. 10 shows that output can reach on 63.3% of steady state value in about 101 second that means the design results provides performance error about under 5%. The disadvantage of this synthesis design was offset-value can be achieved in about 550 second. It was too big for temperature controlled system.

IX. CONCLUSION

The conventional mode has settling time at cost of high overshoots, which may, in some cases, reach 20-40%. On the other hand, the settling time of PWM mode can be regulated for desired duration. The PWM mode now contributes two improvement compared with conventional mode: smaller ripple and smaller error steady state. The verification of algorithm design was conducted for time constant about 95.5 seconds. Using MATLAB Simulink, with certain design parameters, the output signal of bang-bang controller was similar with PID output non-linear.

The magnitude of ripple in steady state condition shows that the bang-bang control with delayed feedback provides good dynamic properties, it is precisely calculated using synthesis concept of controller design. Its value is influenced by the ratio of integrator time and different time in feedback path of controller. The easiest way to validate them is by simulation or experimental measurement. But in general, the ripple of PWM mode may occur, when the controlled variable reaches about 63.3 % of the desired value.

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X. REFERENCES